# Technical Notes

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## A Nonlinear Problem in Rigid Body Dynamics

Belur V. Dasarathy*
*Computer Sciences Corporation, Huntsville, Ala.*

### 1. Introduction

In an earlier Note,[1] illustrating the use of differential transformation techniques in the study of certain classes of third-order nonlinear systems, the classical problem of inertial motion of a rigid body rotating about its center of mass (under zero external moments) was considered as an example. Of course this problem had been solved analytically before by classical techniques.[2]

The relatively more complex problem of a rigid body with an external force acting along one of its principal directions, which is not amenable to earlier approaches,[2,4] is considered in this Note. An analytic approach similar to Ref. 1 is seen to be feasible for this problem also, since the only difference is the external force acting along one of the principal axes.

### 2. Analysis

The Euler's equations of motion of a rigid body rotating about its center of mass with a constant external torque acting along one of the principal axes (i.e., the $x$ axis) can be written as[5]

$$A\dot{\omega}_x + (C - B)\omega_y\omega_z = F_x \tag{1}$$

$$B\dot{\omega}_y + (A - C)\omega_z\omega_x = 0 \tag{2}$$

$$C\dot{\omega}_z + (B - A)\omega_x\omega_y = 0 \tag{3}$$

$(\dot{\ }) \triangleq d(\ )/dt$; time $t$ is the independent variable. Here $A$, $B$, and $C$ are the three principal moments of inertia along the the $x$, $y$, and $z$ axes, respectively, with $\omega_x$, $\omega_y$, $\omega_z$ being the corresponding angular velocities. $F_x$ is the external force along the $x$ axis. Equations (1–3), being *coupled* nonlinear differential equations, are not amenable to the usual methods of analysis in their present form.

Differentiation of Eq. (1) and use of Eqs. (2) and (3) gives

$$A\ddot{\omega}_x - (B - C)\omega_x[C(C - A)\omega_z^2 + B(A - B)\omega_y^2]/BC = 0 \tag{4}$$

Further differentiation of Eq. (4) and use of Eqs. (1–4) leads to a single third-order nonlinear uncoupled differential equation of the form

$$d\ddot{\omega}_x/dt - \dot{\omega}_x\ddot{\omega}_x/\omega_x - D^2\omega_x^2\dot{\omega}_x + D^2F_x\omega_x^2/A = 0 \tag{5a}$$

where

$$D^2 = 4(A - B)(C - A)/BC \tag{5b}$$

Following the technique suggested in Ref. 1, the necessary nonlinear transformations for the dependent and independent variables are

$$X = \omega_x^2/2 \quad T = \int_0^t \omega_x d\tau + \mathrm{K} \tag{6, 7}$$

($\tau$ is a dummy variable) i.e.,

$$X' = \dot{\omega}_x \tag{8a}$$

$[(\ )' \triangleq d(\ )/dT$; $T$ is the new independent variable$]$

$$X'' = \ddot{\omega}_x/\omega_x \tag{8b}$$

$$X''' = (d\ddot{\omega}_x/dt)/\omega_x^2 - \dot{\omega}_x\ddot{\omega}_x/\omega_x^3 \tag{8c}$$

Use of Eqs. (6–8) in Eq. (5) gives

$$X''' - D^2X' + D^2F_x/A = 0 \tag{9}$$

Equation (9) is a linear third-order differential equation with constant coefficients and can be solved by standard techniques to obtain

$$X(T) = c_1 \exp(DT) + c_2 \exp(-DT) + F_xT/A + c_3 = \omega_x^2(T)/2 \tag{10}$$

Rewriting Eq. (7) as

$$t = \int dT/\omega_x(T) = f(T) \tag{11}$$

Substituting from (10) in Eq. (11) and integrating numerically or otherwise gives the relation between $t$ and $T$ which along with Eq. (6) gives the response of the system. Notice that if $\omega_x$ goes through zero, then Eq. (11) becomes an improper integral.

### 3. Discussions and Concluding Remarks

A possible extension worthy of consideration is the case where $F_x$, the external force acting on the system, is a general function of time. Following the methodology adopted earlier of differentiating Eq. (1) successively (keeping in mind that $F_x$ is a variable instead of a constant) leads to an equation similar to Eq. (5) with two additional terms $-(\dot{F}_x\dot{\omega}_x/A\omega_x) + \ddot{F}_x/A$ on the right-hand side.

Modifying Eq. (8a) to

$$X_1' = \dot{\omega}_x - F_x/A \tag{12}$$

leads to a third-order linear equation $X_1''' - D^2X_1' = 0$, whose solution can be written down as

$$X_1(T) = c_1 \exp(DT) + c_2 \exp(-DT) + c_3 \tag{13}$$

But the analysis breaks down at this point, since the relationship between $X_1$ and $\omega_x$ is no longer explicit as in Eq. (6) due to the modification made in Eq. (8a) [Eq. (12)]; i.e.,

$$X_1(T) = \omega_x^2/2 - (1/A)\int F_x(t)dT$$

which can be evaluated explicitly only when $F_x(t)$ is a constant, because at this stage the relationship between $t$ and $T$ is unknown.

When $F_x(t)$ is a constant,

$$X_1(T) = \omega_x^2(T)/2 - F_xT/A \tag{14}$$

Combining Eqs. (13) and (14) gives Eq. (10) as before. Thus the case of variable input, although reducible as before to an equivalent linear problem, cannot, even in theoretical terms, be solved to completion. Although alternate approaches to overcome this limitation are under investigation, the method is being presented as a forerunner.

Unlike the classical case considered in Ref. 1, which could be considered in a formal way (see Refs. 3, 4) as a second-order

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* Senior Member Technical Staff. Member AIAA.

system, the present problem represented by Eq. (5) cannot be reduced by integration to a second-order system. Thus this method becomes almost unique for the study of this nonlinear problem in rigid body dynamics.


### References

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# Changes in Heat Transfer from Turbulent Boundary Layers Interacting with Shock Waves and Expansion Waves


L. H. BACK* AND R. F. CUFFEL†

*Jet Propulsion Laboratory, Pasadena, California*


### Nomenclature

$a^*$ = speed of sound at sonic condition
$c$ = constant
$h$ = heat transfer coefficient
$H$ = enthalpy
$M$ = Mach number
$p$ = pressure
$Pr$ = Prandtl number
$q$ = heat flux to wall
$r$ = channel radius
$Re$/ft = Reynolds number per ft, $\rho_e u_e/\mu_e$
$St$ = Stanton number
$T$ = temperature
$u$ = velocity parallel to wall
$x$ = distance along wall
$z$ = axial distance
$\gamma$ = specific heat ratio
$\mu$ = viscosity
$\rho$ = density
$\sigma$ = flow deflection angle
$\phi$ = energy thickness
$\omega$ = viscosity-temperature exponent

***Subscripts***

$aw$ = adiabatic wall condition
$e$ = freestream condition
$0$ = reservoir condition
$t$ = stagnation condition
$w$ = wall condition
$1$ = upstream value
$2$ = downstream or peak value


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* Member Technical Staff, Propulsion Research and Advanced Concepts Section. Associate Fellow AIAA.

† Senior Engineer, Propulsion Research and Advanced Concepts Section. Member AIAA.


### Introduction

THIS Note is concerned with heat transfer from turbulent boundary layers in supersonic flows where changes in surface curvature can produce shock waves and expansion waves, e.g., corner flows, or where shock waves generated elsewhere in the flow impinge on the boundary layer. Heat-transfer measurements are presented in these interaction regions and a rather simple method involving the integral form of the energy equation is used to estimate the change in heat transfer that is observed. The prediction is then compared to other experimental data obtained at shock impingement locations.

### Measurements

The interaction regions investigated were observed in an axisymmetric supersonic diffuser with air at stagnation pressures of 100, 150, and 200 psia and a stagnation temperature of 1500°R. The measurements included wall static pressures, semilocal wall heat fluxes, coolant-side wall temperatures, and boundary-layer surveys. The heat flux was determined by calorimetric measurements in circumferential coolant passages. The gas-side wall temperature was calculated from the measured wall heat flux and coolant-side wall temperature. A nearly isothermal wall condition was achieved with a wall-to-stagnation temperature ratio of ~0.43. The heat-transfer coefficients, calculated by using the difference between adiabatic wall enthalpy and wall enthalpy, are believed accurate to ~10%. The adiabatic wall enthalpy was calculated using a recovery factor of 0.89, a value which does not seem to differ much in shock impingement regions.[1,2] Reference 3 describes the test apparatus and Ref. 4 the measurement technique as applied to flow through a nozzle.

Boundary-layer surveys were made with a small flattened Pitot tube and an aspirating thermocouple probe with a recovery factor of 0.97. The tips of these probes were 0.005 and 0.010 in. high, respectively, relatively small compared to the boundary-layer thickness. Only the turbulent boundary-layer thicknesses are shown, not the profiles themselves.

### Results

Measurements in an expanding flow around a 6° corner and at a shock wave impingement on a turbulent boundary layer are shown in Fig. 1. The expansion fan emanating from the corner is drawn for a uniform, inviscid flow and the nature of the interaction in the impingement region is shown diagrammatically. After impingement, the reflected shock wave is curved away from the wall and this leads to an expanding flow downstream as indicated by the decreasing wall pressures. For the corner flow, a local Prandtl-Meyer expansion at the corner gives a downstream Mach number of 3.59, in close agreement with the measured value. However, a calculation

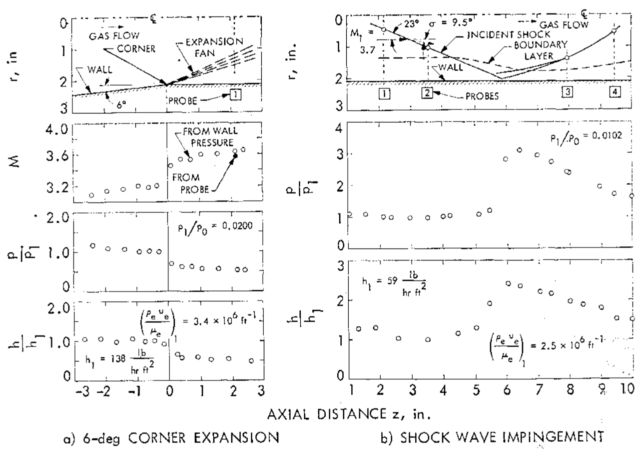


**Fig. 1 Measurements in interaction regions, $p_{t0} = 100.5$ psia, $T_{t0} = 1500°$R, $T_w/T_{t0} = 0.42$–$0.45$.**